Strategy for Integration

As we have seen, integration is more challenging than differentiation. In finding the derivative of a function it is obvious which differentiation formula we should apply. But it may not be obvious which technique we should use to integrate a given function.

Until now individual techniques have been applied in each section. For instance, we usually used substitution in Exercises 5.5, integration by parts in Exercises 5.6, and partial fractions in Exercises 5.7 and Appendix G. But in this section we present a collection of miscellaneous integrals in random order and the main challenge is to recognize which technique or formula to use. No hard and fast rules can be given as to which method applies in a given situation, but we give some advice on strategy that you may find useful.

A prerequisite for strategy selection is a knowledge of the basic integration formulas. In the following table we have collected the integrals from our previous list together with several additional formulas that we have learned in this chapter. Most of them should be memorized. It is useful to know them all, but the ones marked with an asterisk need not be memorized since they are easily derived. Formula 19 can be avoided by using partial fractions, and trigonometric substitutions can be used in place of Formula 20.

Table of Integration Formulas Constants of integration have been omitted.

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$ **2.** $\displaystyle\int \frac{1}{x}\,dx = \ln|x|$

3. $\displaystyle\int e^x\,dx = e^x$ **4.** $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a}$

5. $\displaystyle\int \sin x\,dx = -\cos x$ **6.** $\displaystyle\int \cos x\,dx = \sin x$

7. $\displaystyle\int \sec^2 x\,dx = \tan x$ **8.** $\displaystyle\int \csc^2 x\,dx = -\cot x$

9. $\displaystyle\int \sec x \tan x\,dx = \sec x$ **10.** $\displaystyle\int \csc x \cot x\,dx = -\csc x$

11. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$ **12.** $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x|$

13. $\displaystyle\int \tan x\,dx = \ln|\sec x|$ **14.** $\displaystyle\int \cot x\,dx = \ln|\sin x|$

15. $\displaystyle\int \sinh x\,dx = \cosh x$ **16.** $\displaystyle\int \cosh x\,dx = \sinh x$

17. $\displaystyle\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ **18.** $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

***19.** $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$ ***20.** $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Once you are armed with these basic integration formulas, if you don't immediately see how to attack a given integral, you might try the following four-step strategy.

1. Simplify the Integrand if Possible Sometimes the use of algebraic manipulation or trigonometric identities will simplify the integrand and make the method of integration obvious. Here are some examples:

$$\int \sqrt{x}\,(1 + \sqrt{x})\,dx = \int (\sqrt{x} + x)\,dx$$

$$\int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\cos^2\theta\,d\theta$$

$$= \int \sin\theta\cos\theta\,d\theta = \tfrac{1}{2}\int \sin 2\theta\,d\theta$$

$$\int (\sin x + \cos x)^2\,dx = \int (\sin^2 x + 2\sin x\cos x + \cos^2 x)\,dx$$

$$= \int (1 + 2\sin x\cos x)\,dx$$

2. Look for an Obvious Substitution Try to find some function $u = g(x)$ in the integrand whose differential $du = g'(x)\,dx$ also occurs, apart from a constant factor. For instance, in the integral

$$\int \frac{x}{x^2 - 1}\,dx$$

we notice that if $u = x^2 - 1$, then $du = 2x\,dx$. Therefore, we use the substitution $u = x^2 - 1$ instead of the method of partial fractions.

3. Classify the Integrand According to Its Form If Steps 1 and 2 have not led to the solution, then we take a look at the form of the integrand $f(x)$.

(a) *Trigonometric functions.* If $f(x)$ is a product of powers of $\sin x$ and $\cos x$, of $\tan x$ and $\sec x$, or of $\cot x$ and $\csc x$, then we use the substitutions recommended in Section 5.7 and *Additional Topics: Trigonometric Integrals*.

(b) *Rational functions.* If f is a rational function, we use the procedure involving partial fractions in Section 5.7 and Appendix G.

(c) *Integration by parts.* If $f(x)$ is a product of a power of x (or a polynomial) and a transcendental function (such as a trigonometric, exponential, or logarithmic function), then we try integration by parts, choosing u and dv according to the advice given in Section 5.6. If you look at the functions in Exercises 5.6, you will see that most of them are the type just described.

(d) *Radicals.* Particular kinds of substitutions are recommended when certain radicals appear.

(i) If $\sqrt{\pm x^2 \pm a^2}$ occurs, we use a trigonometric substitution according to the table in *Additional Topics: Trigonometric Substitution*.

(ii) If $\sqrt[n]{ax + b}$ occurs, we use the rationalizing substitution $u = \sqrt[n]{ax + b}$. More generally, this sometimes works for $\sqrt[n]{g(x)}$.

4. Try Again If the first three steps have not produced the answer, remember that there are basically only two methods of integration: substitution and parts.

(a) *Try substitution.* Even if no substitution is obvious (Step 2), some inspiration or ingenuity (or even desperation) may suggest an appropriate substitution.

(b) *Try parts.* Although integration by parts is used most of the time on products of the form described in Step 3(c), it is sometimes effective on single functions. Looking at Section 5.6, we see that it works on $\tan^{-1}x$, $\sin^{-1}x$, and $\ln x$, and these are all inverse functions.

(c) *Manipulate the integrand.* Algebraic manipulations (perhaps rationalizing the denominator or using trigonometric identities) may be useful in transforming the integral into an easier form. These manipulations may be more substantial than in Step 1 and may involve some ingenuity. Here is an example:

$$\int \frac{dx}{1 - \cos x} = \int \frac{1}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x}\, dx = \int \frac{1 + \cos x}{1 - \cos^2 x}\, dx$$

$$= \int \frac{1 + \cos x}{\sin^2 x}\, dx = \int \left(\csc^2 x + \frac{\cos x}{\sin^2 x} \right) dx$$

(d) *Relate the problem to previous problems.* When you have built up some experience in integration, you may be able to use a method on a given integral that is similar to a method you have already used on a previous integral. Or you may even be able to express the given integral in terms of a previous one. For instance, $\int \tan^2 x \sec x\, dx$ is a challenging integral, but if we make use of the identity $\tan^2 x = \sec^2 x - 1$, we can write

$$\int \tan^2 x \sec x\, dx = \int \sec^3 x\, dx - \int \sec x\, dx$$

and if $\int \sec^3 x\, dx$ has previously been evaluated (see Example 8 in *Additional Topics: Trigonometric Integrals*), then that calculation can be used in the present problem.

(e) *Use several methods.* Sometimes two or three methods are required to evaluate an integral. The evaluation could involve several successive substitutions of different types, or it might combine integration by parts with one or more substitutions.

In the following examples we indicate a method of attack but do not fully work out the integral.

EXAMPLE 1 $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\, dx$

In Step 1 we rewrite the integral:

$$\int \frac{\tan^3 x}{\cos^3 x}\, dx = \int \tan^3 x \sec^3 x\, dx$$

The integral is now of the form $\int \tan^m x \sec^n x\, dx$ with m odd, so we can use the advice in *Additional Topics: Trigonometric Integrals*.

Alternatively, if in Step 1 we had written

$$\int \frac{\tan^3 x}{\cos^3 x}\, dx = \int \frac{\sin^3 x}{\cos^3 x} \frac{1}{\cos^3 x}\, dx = \int \frac{\sin^3 x}{\cos^6 x}\, dx$$

then we could have continued as follows with the substitution $u = \cos x$:

$$\int \frac{\sin^3 x}{\cos^6 x}\, dx = \int \frac{1 - \cos^2 x}{\cos^6 x} \sin x\, dx = \int \frac{1 - u^2}{u^6} (-du)$$

$$= \int \frac{u^2 - 1}{u^6}\, du = \int (u^{-4} - u^{-6})\, du$$

■■

EXAMPLE 2 $\displaystyle\int e^{\sqrt{x}}\, dx$

According to Step 3(d)(ii) we substitute $u = \sqrt{x}$. Then $x = u^2$, so $dx = 2u\, du$ and

$$\int e^{\sqrt{x}}\, dx = 2 \int u e^u\, du$$

The integrand is now a product of u and the transcendental function e^u so it can be integrated by parts. ■■

EXAMPLE 3 $\displaystyle\int \frac{x^5 + 1}{x^3 - 3x^2 - 10x}\,dx$

No algebraic simplification or substitution is obvious, so Steps 1 and 2 don't apply here. The integrand is a rational function so we apply the procedure of Appendix G, remembering that the first step is to divide. ■■

EXAMPLE 4 $\displaystyle\int \frac{dx}{x\sqrt{\ln x}}$

Here Step 2 is all that is needed. We substitute $u = \ln x$ because its differential is $du = dx/x$, which occurs in the integral. ■■

EXAMPLE 5 $\displaystyle\int \sqrt{\frac{1 - x}{1 + x}}\,dx$

Although the rationalizing substitution

$$u = \sqrt{\frac{1 - x}{1 + x}}$$

works here [Step 3(d)(ii)], it leads to a very complicated rational function. An easier method is to do some algebraic manipulation [either as Step 1 or as Step 4(c)]. Multiplying numerator and denominator by $\sqrt{1 - x}$, we have

$$\begin{aligned}\int \sqrt{\frac{1 - x}{1 + x}}\,dx &= \int \frac{1 - x}{\sqrt{1 - x^2}}\,dx \\ &= \int \frac{1}{\sqrt{1 - x^2}}\,dx - \int \frac{x}{\sqrt{1 - x^2}}\,dx \\ &= \sin^{-1}x + \sqrt{1 - x^2} + C\end{aligned}$$

■■

Can We Integrate All Continuous Functions?

The question arises: Will our strategy for integration enable us to find the integral of every continuous function? For example, can we use it to evaluate $\int e^{x^2}\,dx$? The answer is no, at least not in terms of the functions that we are familiar with.

The functions that we have been dealing with in this book are called **elementary functions**. These are the polynomials, rational functions, power functions (x^a), exponential functions (a^x), logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition. For instance, the function

$$f(x) = \sqrt{\frac{x^2 - 1}{x^3 + 2x - 1}} + \ln(\cosh x) - xe^{\sin 2x}$$

is an elementary function.

If f is an elementary function, then f' is an elementary function but $\int f(x)\,dx$ need not be an elementary function. Consider $f(x) = e^{x^2}$. Since f is continuous, its integral exists, and if we define the function F by

$$F(x) = \int_0^x e^{t^2}\,dt$$

then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^{x^2}$$

Thus, $f(x) = e^{x^2}$ has an antiderivative F, but it has been proved that F is not an elementary function. This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2}\,dx$ in terms of the functions we know. (In Chapter 8, however, we will see how to express $\int e^{x^2}\,dx$ as an infinite series.) The same can be said of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3+1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions don't have elementary antiderivatives. You may be assured, though, that the integrals in the following exercises are all elementary functions.

Exercises

A Click here for answers.

S Click here for solutions.

1–80 ■ Evaluate the integral.

1. $\int \frac{\sin x + \sec x}{\tan x}\,dx$
2. $\int \tan^3\theta\,d\theta$
3. $\int_0^2 \frac{2t}{(t-3)^2}\,dt$
4. $\int \frac{x}{\sqrt{3-x^4}}\,dx$
5. $\int_{-1}^1 \frac{e^{\arctan y}}{1+y^2}\,dy$
6. $\int x\csc x\,\cot x\,dx$
7. $\int_1^3 r^4\ln r\,dr$
8. $\int_0^4 \frac{x-1}{x^2-4x-5}\,dx$
9. $\int \frac{x-1}{x^2-4x+5}\,dx$
10. $\int \frac{x}{x^4+x^2+1}\,dx$
11. $\int \sin^3\theta\,\cos^5\theta\,d\theta$
12. $\int \sin x\,\cos(\cos x)\,dx$
13. $\int \frac{dx}{(1-x^2)^{3/2}}$
14. $\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx$
15. $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$
16. $\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$
17. $\int x\sin^2 x\,dx$
18. $\int \frac{e^{2t}}{1+e^{4t}}\,dt$
19. $\int e^{x+e^x}\,dx$
20. $\int e^{\sqrt[3]{x}}\,dx$
21. $\int t^3e^{-2t}\,dt$
22. $\int x\sin^{-1}x\,dx$
23. $\int_0^1 (1+\sqrt{x})^8\,dx$
24. $\int \ln(x^2-1)\,dx$
25. $\int \frac{3x^2-2}{x^2-2x-8}\,dx$
26. $\int \frac{3x^2-2}{x^3-2x-8}\,dx$
27. $\int \cot x\,\ln(\sin x)\,dx$
28. $\int \sin\sqrt{at}\,dt$
29. $\int_0^5 \frac{3w-1}{w+2}\,dw$
30. $\int_{-2}^2 |x^2-4x|\,dx$
31. $\int \sqrt{\frac{1+x}{1-x}}\,dx$
32. $\int \frac{\sqrt{2x-1}}{2x+3}\,dx$
33. $\int \sqrt{3-2x-x^2}\,dx$
34. $\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx$
35. $\int_{-1}^1 x^8\sin x\,dx$
36. $\int \sin 4x\,\cos 3x\,dx$
37. $\int_0^{\pi/4} \cos^2\theta\,\tan^2\theta\,d\theta$
38. $\int_0^{\pi/4} \tan^5\theta\,\sec^3\theta\,d\theta$
39. $\int \frac{x}{1-x^2+\sqrt{1-x^2}}\,dx$
40. $\int \frac{1}{\sqrt{4y^2-4y-3}}\,dy$
41. $\int \theta\tan^2\theta\,d\theta$
42. $\int x^2\tan^{-1}x\,dx$
43. $\int e^x\sqrt{1+e^x}\,dx$
44. $\int \sqrt{1+e^x}\,dx$
45. $\int x^5e^{-x^3}\,dx$
46. $\int \frac{1+e^x}{1-e^x}\,dx$
47. $\int \frac{x+a}{x^2+a^2}\,dx$
48. $\int \frac{x}{x^4-a^4}\,dx$
49. $\int \frac{1}{x\sqrt{4x+1}}\,dx$
50. $\int \frac{1}{x^2\sqrt{4x+1}}\,dx$
51. $\int \frac{1}{x\sqrt{4x^2+1}}\,dx$
52. $\int \frac{dx}{x(x^4+1)}$

53. $\int x^2 \sinh mx\,dx$

54. $\int (x + \sin x)^2\,dx$

55. $\int \dfrac{1}{x + 4 + 4\sqrt{x + 1}}\,dx$

56. $\int \dfrac{x \ln x}{\sqrt{x^2 - 1}}\,dx$

57. $\int x\sqrt[3]{x + c}\,dx$

58. $\int x^2 \ln(1 + x)\,dx$

59. $\int \dfrac{1}{e^{3x} - e^x}\,dx$

60. $\int \dfrac{1}{x + \sqrt[3]{x}}\,dx$

61. $\int \dfrac{x^4}{x^{10} + 16}\,dx$

62. $\int \dfrac{x^3}{(x + 1)^{10}}\,dx$

63. $\int \sqrt{x}e^{\sqrt{x}}\,dx$

64. $\int_{\pi/4}^{\pi/3} \dfrac{\ln(\tan x)}{\sin x \cos x}\,dx$

65. $\int \dfrac{1}{\sqrt{x + 1} + \sqrt{x}}\,dx$

66. $\int_2^3 \dfrac{u^3 + 1}{u^3 - u^2}\,du$

67. $\int_1^3 \dfrac{\arctan\sqrt{t}}{\sqrt{t}}\,dt$

68. $\int \dfrac{1}{1 + 2e^x - e^{-x}}\,dx$

69. $\int \dfrac{e^{2x}}{1 + e^x}\,dx$

70. $\int \dfrac{\ln(x + 1)}{x^2}\,dx$

71. $\int \dfrac{x}{x^4 + 4x^2 + 3}\,dx$

72. $\int \dfrac{\sqrt{t}}{1 + \sqrt[3]{t}}\,dt$

73. $\int \dfrac{1}{(x - 2)(x^2 + 4)}\,dx$

74. $\int \dfrac{dx}{e^x - e^{-x}}$

75. $\int \sin x \sin 2x \sin 3x\,dx$

76. $\int (x^2 - bx) \sin 2x\,dx$

77. $\int \dfrac{\sqrt{x}}{1 + x^3}\,dx$

78. $\int \dfrac{\sec x \cos 2x}{\sin x + \sec x}\,dx$

79. $\int x \sin^2 x \cos x\,dx$

80. $\int \dfrac{\sin x \cos x}{\sin^4 x + \cos^4 x}\,dx$

81. The functions $y = e^{x^2}$ and $y = x^2 e^{x^2}$ don't have elementary antiderivatives, but $y = (2x^2 + 1)e^{x^2}$ does. Evaluate $\int (2x^2 + 1)e^{x^2}\,dx$.

Answers

S Click here for solutions.

1. $\sin x + \ln|\csc x - \cot x| + C$
3. $4 - \ln 9$ **5.** $e^{\pi/4} - e^{-\pi/4}$
7. $\frac{243}{5}\ln 3 - \frac{242}{25}$ **9.** $\frac{1}{2}\ln(x^2 - 4x + 5) + \tan^{-1}(x-2) + C$
11. $\frac{1}{8}\cos^8\theta - \frac{1}{6}\cos^6\theta + C$ $\left(\text{or } \frac{1}{4}\sin^4\theta - \frac{1}{3}\sin^6\theta + \frac{1}{8}\sin^8\theta + C\right)$
13. $x/\sqrt{1-x^2} + C$ **15.** $1 - \frac{1}{2}\sqrt{3}$
17. $\frac{1}{4}x^2 - \frac{1}{2}x\sin x\cos x + \frac{1}{4}\sin^2 x + C$
$\left(\text{or } \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C\right)$
19. $e^{e^x} + C$ **21.** $-\frac{1}{8}e^{-2t}(4t^3 + 6t^2 + 6t + 3) + C$
23. $\frac{4097}{45}$ **25.** $3x + \frac{23}{3}\ln|x-4| - \frac{5}{3}\ln|x+2| + C$
27. $\frac{1}{2}(\ln\sin x)^2 + C$ **29.** $15 + 7\ln\frac{2}{7}$
31. $\sin^{-1}x - \sqrt{1-x^2} + C$
33. $2\sin^{-1}((x+1)/2) + ((x+1)/2)\sqrt{3 - 2x - x^2} + C$
35. 0 **37.** $\pi/8 - \frac{1}{4}$ **39.** $-\ln\left(1 + \sqrt{1-x^2}\right) + C$
41. $\theta\tan\theta - \frac{1}{2}\theta^2 - \ln|\sec\theta| + C$ **43.** $\frac{2}{3}(1+e^x)^{3/2} + C$
45. $-\frac{1}{3}(x^3+1)e^{-x^3} + C$
47. $\ln\sqrt{x^2+a^2} + \tan^{-1}(x/a) + C$
49. $\ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$ **51.** $-\ln\left|\frac{\sqrt{4x^2+1}+1}{2x}\right| + C$
53. $(1/m)x^2\cosh(mx) - (2/m^2)x\sinh(mx) + (2/m^3)\cosh(mx) + C$
55. $3\ln\left(\sqrt{x+1}+3\right) - \ln\left(\sqrt{x+1}+1\right) + C$
57. $\frac{3}{7}(x+c)^{7/3} - \frac{3}{4}c(x+c)^{4/3} + C$
59. $e^{-x} + \frac{1}{2}\ln|(e^x-1)/(e^x+1)| + C$
61. $\frac{1}{20}\tan^{-1}\left(\frac{1}{4}x^5\right) + C$ **63.** $2\left(x - 2\sqrt{x} + 2\right)e^{\sqrt{x}} + C$
65. $\frac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C$
67. $\frac{2}{3}\sqrt{3}\pi - \frac{1}{2}\pi - \ln 2$ **69.** $e^x - \ln(1+e^x) + C$
71. $-\frac{1}{4}\ln(x^2+3) + \frac{1}{4}\ln(x^2+1) + C$
73. $\frac{1}{8}\ln|x-2| - \frac{1}{16}\ln(x^2+4) - \frac{1}{8}\tan^{-1}(x/2) + C$
75. $\frac{1}{24}\cos 6x - \frac{1}{16}\cos 4x - \frac{1}{8}\cos 2x + C$ **77.** $\frac{2}{3}\tan^{-1}(x^{3/2}) + C$
79. $\frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x + C$ **81.** $xe^{x^2} + C$

Solutions: Strategy for Integration

1. $\displaystyle\int \frac{\sin x + \sec x}{\tan x}\,dx = \int \left(\frac{\sin x}{\tan x} + \frac{\sec x}{\tan x}\right) dx = \int (\cos x + \csc x)\,dx = \sin x + \ln|\csc x - \cot x| + C$

3. $\displaystyle\int_0^2 \frac{2t}{(t-3)^2}\,dt = \int_{-3}^{-1} \frac{2\,(u+3)}{u^2}\,du \;\; [u = t-3,\, du = dt] \;\; = \int_{-3}^{-1}\left(\frac{2}{u} + \frac{6}{u^2}\right) du = \left[2\ln|u| - \frac{6}{u}\right]_{-3}^{-1}$

$\displaystyle = (2\ln 1 + 6) - (2\ln 3 + 2) = 4 - 2\ln 3 \text{ or } 4 - \ln 9$

5. Let $u = \arctan y$. Then $\displaystyle du = \frac{dy}{1+y^2} \;\Rightarrow\; \int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = [e^u]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}.$

7. $\displaystyle\int_1^3 r^4 \ln r\,dr \;\begin{bmatrix} u = \ln r, & dv = r^4\,dr, \\ du = \dfrac{dr}{r} & v = \dfrac{1}{5}r^5 \end{bmatrix} = \left[\frac{1}{5}r^5 \ln r\right]_1^3 - \int_1^3 \frac{1}{5}r^4\,dr = \frac{243}{5}\ln 3 - 0 - \left[\frac{1}{25}r^5\right]_1^3$

$\displaystyle = \tfrac{243}{5}\ln 3 - \left(\tfrac{243}{25} - \tfrac{1}{25}\right) = \tfrac{243}{5}\ln 3 - \tfrac{242}{25}$

9. $\displaystyle\int \frac{x-1}{x^2 - 4x + 5}\,dx = \int \frac{(x-2)+1}{(x-2)^2+1}\,dx = \int \left(\frac{u}{u^2+1} + \frac{1}{u^2+1}\right) du \quad [u = x-2,\, du = dx]$

$\displaystyle = \tfrac{1}{2}\ln\left(u^2+1\right) + \tan^{-1} u + C = \tfrac{1}{2}\ln\left(x^2 - 4x + 5\right) + \tan^{-1}(x-2) + C$

11. $\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \cos^5\theta \sin^2\theta \sin\theta\,d\theta = -\int \cos^5\theta\,(1 - \cos^2\theta)(-\sin\theta)\,d\theta$

$\displaystyle = -\int u^5(1-u^2)\,du \quad \begin{bmatrix} u = \cos\theta, \\ du = -\sin\theta\,d\theta \end{bmatrix}$

$= \int (u^7 - u^5)\,du = \frac{1}{8}u^8 - \frac{1}{6}u^6 + C = \frac{1}{8}\cos^8\theta - \frac{1}{6}\cos^6\theta + C$

Another solution:

$\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \sin^3\theta\,(\cos^2\theta)^2 \cos\theta\,d\theta = \int \sin^3\theta\,(1 - \sin^2\theta)^2 \cos\theta\,d\theta$

$\displaystyle = \int u^3(1-u^2)^2\,du \quad \begin{bmatrix} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{bmatrix} \quad = \int u^3(1 - 2u^2 + u^4)\,du$

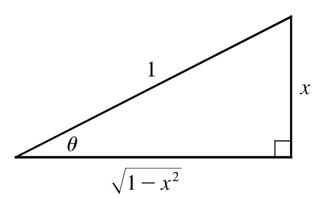
$= \int (u^3 - 2u^5 + u^7)\,du = \frac{1}{4}u^4 - \frac{1}{3}u^6 + \frac{1}{8}u^8 + C = \frac{1}{4}\sin^4\theta - \frac{1}{3}\sin^6\theta + \frac{1}{8}\sin^8\theta + C$

13. Let $x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = \cos\theta\,d\theta$ and

$(1 - x^2)^{1/2} = \cos\theta$, so

$\displaystyle\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta\,d\theta}{(\cos\theta)^3} = \int \sec^2\theta\,d\theta$

$\displaystyle = \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C$



15. Let $u = 1 - x^2 \;\Rightarrow\; du = -2x\,dx$. Then

$\displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_{3/4}^{1} u^{-1/2}\,du = \tfrac{1}{2}\left[2u^{1/2}\right]_{3/4}^{1} = \left[\sqrt{u}\right]_{3/4}^{1} = 1 - \tfrac{\sqrt{3}}{2}$

17. $\displaystyle\int x\sin^2 x\,dx \quad \begin{bmatrix} u = x, & dv = \sin^2 x\,dx, \\ du = dx & v = \int \sin^2 x\,dx = \int \tfrac{1}{2}(1 - \cos 2x)\,dx = \tfrac{1}{2}x - \tfrac{1}{2}\sin x\cos x \end{bmatrix}$

$= \frac{1}{2}x^2 - \frac{1}{2}x\sin x\cos x - \int \left(\frac{1}{2}x - \frac{1}{2}\sin x\cos x\right) dx$

$= \frac{1}{2}x^2 - \frac{1}{2}x\sin x\cos x - \frac{1}{4}x^2 + \frac{1}{4}\sin^2 x + C = \frac{1}{4}x^2 - \frac{1}{2}x\sin x\cos x + \frac{1}{4}\sin^2 x + C$

Note: $\int \sin x\cos x\,dx = \int s\,ds = \frac{1}{2}s^2 + C \quad [\text{where } s = \sin x,\, ds = \cos x\,dx]$.

A slightly different method is to write $\int x\sin^2 x\,dx = \int x \cdot \frac{1}{2}(1 - \cos 2x)\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx$. If we evaluate the second integral by parts, we arrive at the equivalent answer $\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$.

19. Let $u = e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x} e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$.

21. Integrate by parts three times, first with $u = t^3$, $dv = e^{-2t}\,dt$:

$$\int t^3 e^{-2t}\,dt = -\tfrac{1}{2}t^3 e^{-2t} + \tfrac{1}{2}\int 3t^2 e^{-2t}\,dt = -\tfrac{1}{2}t^3 e^{-2t} - \tfrac{3}{4}t^2 e^{-2t} + \tfrac{1}{2}\int 3te^{-2t}\,dt$$

$$= -e^{-2t}\left[\tfrac{1}{2}t^3 + \tfrac{3}{4}t^2\right] - \tfrac{3}{4}te^{-2t} + \tfrac{3}{4}\int e^{-2t}\,dt = -e^{-2t}\left[\tfrac{1}{2}t^3 + \tfrac{3}{4}t^2 + \tfrac{3}{4}t + \tfrac{3}{8}\right] + C$$

$$= -\tfrac{1}{8}e^{-2t}\left(4t^3 + 6t^2 + 6t + 3\right) + C$$

23. Let $u = 1 + \sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1)\,du \quad\Rightarrow$

$$\int_0^1 \left(1+\sqrt{x}\right)^8\,dx = \int_1^2 u^8 \cdot 2(u-1)\,du = 2\int_1^2 \left(u^9 - u^8\right)du = \left[\tfrac{1}{5}u^{10} - 2\cdot\tfrac{1}{9}u^9\right]_1^2$$

$$= \tfrac{1024}{5} - \tfrac{1024}{9} - \tfrac{1}{5} + \tfrac{2}{9} = \tfrac{4097}{45}$$

25. $\dfrac{3x^2-2}{x^2-2x-8} = 3 + \dfrac{6x+22}{(x-4)(x+2)} = 3 + \dfrac{A}{x-4} + \dfrac{B}{x+2} \quad\Rightarrow\quad 6x + 22 = A(x+2) + B(x-4)$. Setting $x = 4$ gives $46 = 6A$, so $A = \frac{23}{3}$. Setting $x = -2$ gives $10 = -6B$, so $B = -\frac{5}{3}$. Now

$$\int \frac{3x^2-2}{x^2-2x-8}\,dx = \int \left(3 + \frac{23/3}{x-4} - \frac{5/3}{x+2}\right)dx = 3x + \tfrac{23}{3}\ln|x-4| - \tfrac{5}{3}\ln|x+2| + C.$$

27. Let $u = \ln(\sin x)$. Then $du = \cot x\,dx \quad\Rightarrow\quad \int \cot x \ln(\sin x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}[\ln(\sin x)]^2 + C$.

29. $$\int_0^5 \frac{3w-1}{w+2}\,dw = \int_0^5 \left(3 - \frac{7}{w+2}\right)dw = \Big[3w - 7\ln|w+2|\Big]_0^5$$

$$= 15 - 7\ln 7 + 7\ln 2 = 15 + 7(\ln 2 - \ln 7) = 15 + 7\ln\tfrac{2}{7}$$

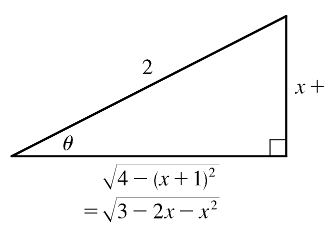
31. As in Example 5,

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x - \sqrt{1-x^2} + C$$

Another method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

33. $3 - 2x - x^2 = -(x^2 + 2x + 1) + 4 = 4 - (x+1)^2$. Let $x + 1 = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\,d\theta$ and



$$\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx = \int \sqrt{4 - 4\sin^2\theta}\,2\cos\theta\,d\theta$$

$$= 4\int \cos^2\theta\,d\theta = 2\int (1 + \cos 2\theta)\,d\theta$$

$$= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2\cdot\frac{x+1}{2}\cdot\frac{\sqrt{3-2x-x^2}}{2} + C$$

$$= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3-2x-x^2} + C$$

35. Because $f(x) = x^8 \sin x$ is the product of an even function and an odd function, it is odd. Therefore, $\int_{-1}^1 x^8 \sin x\,dx = 0 \quad$ [by (5.5.7)(b)].

37. $\int_0^{\pi/4} \cos^2\theta \tan^2\theta \, d\theta = \int_0^{\pi/4} \sin^2\theta \, d\theta = \int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta)\, d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/4}$
$= \left(\frac{\pi}{8} - \frac{1}{4}\right) - (0-0) = \frac{\pi}{8} - \frac{1}{4}$

39. Let $u = 1 - x^2$. Then $du = -2x\,dx \Rightarrow$

$$\int \frac{x\,dx}{1 - x^2 + \sqrt{1-x^2}} = -\frac{1}{2}\int \frac{du}{u + \sqrt{u}} = -\int \frac{v\,dv}{v^2 + v} \quad [v = \sqrt{u},\, u = v^2,\, du = 2v\,dv]$$
$$= -\int \frac{dv}{v+1} = -\ln|v+1| + C = -\ln\left(\sqrt{1-x^2} + 1\right) + C$$

41. Let $u = \theta$, $dv = \tan^2\theta\,d\theta = \left(\sec^2\theta - 1\right)d\theta \Rightarrow du = d\theta$ and $v = \tan\theta - \theta$. So

$$\int \theta\tan^2\theta\,d\theta = \theta(\tan\theta - \theta) - \int(\tan\theta - \theta)\,d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C$$
$$= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C$$

43. Let $u = 1 + e^x$, so that $du = e^x\,dx$. Then
$\int e^x\sqrt{1+e^x}\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C.$
Or: Let $u = \sqrt{1+e^x}$, so that $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$. Then
$\int e^x\sqrt{1+e^x}\,dx = \int u\cdot 2u\,du = \int 2u^2 du = \frac{2}{3}u^3 + C = \frac{2}{3}(1+e^x)^{3/2} + C.$

45. Let $t = x^3$. Then $dt = 3x^2\,dx \Rightarrow I = \int x^5 e^{-x^3}\,dx = \frac{1}{3}\int te^{-t}\,dt$. Now integrate by parts with $u = t$,
$dv = e^{-t}\,dt$: $I = -\frac{1}{3}te^{-t} + \frac{1}{3}\int e^{-t}\,dt = -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + C = -\frac{1}{3}e^{-x^3}\left(x^3+1\right) + C.$

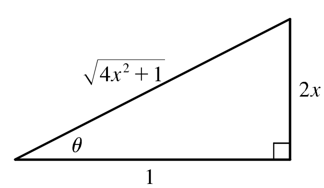
47. $$\int \frac{x+a}{x^2+a^2}\,dx = \frac{1}{2}\int \frac{2x\,dx}{x^2+a^2} + a\int\frac{dx}{x^2+a^2} = \tfrac{1}{2}\ln\left(x^2+a^2\right) + a\cdot\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$
$$= \ln\sqrt{x^2+a^2} + \tan^{-1}(x/a) + C$$

49. Let $u = \sqrt{4x+1} \Rightarrow u^2 = 4x+1 \Rightarrow 2u\,du = 4\,dx \Rightarrow dx = \frac{1}{2}u\,du$. So

$$\int \frac{1}{x\sqrt{4x+1}}\,dx = \int \frac{\frac{1}{2}u\,du}{\frac{1}{4}(u^2-1)\,u} = 2\int\frac{du}{u^2-1} = 2\left(\tfrac{1}{2}\right)\ln\left|\frac{u-1}{u+1}\right| + C \quad \text{[by Formula 19]}$$
$$= \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$$

51. Let $2x = \tan\theta \Rightarrow x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\,d\theta$, $\sqrt{4x^2+1} = \sec\theta$, so

$$\int\frac{dx}{x\sqrt{4x^2+1}} = \int\frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{2}\tan\theta\sec\theta} = \int\frac{\sec\theta}{\tan\theta}\,d\theta = \int\csc\theta\,d\theta$$
$$= -\ln|\csc\theta + \cot\theta| + C \qquad [\text{or } \ln|\csc\theta - \cot\theta| + C]$$
$$= -\ln\left|\frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x}\right| + C \qquad \left[\text{or } \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + C\right]$$



53. $$\int x^2\sinh(mx)dx = \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\int x\cosh(mx)\,dx \qquad \begin{bmatrix} u = x^2, & dv = \sinh(mx)\,dx, \\ du = 2x\,dx & v = \frac{1}{m}\cosh(mx)\end{bmatrix}$$
$$= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\left(\tfrac{1}{m}x\sinh(mx) - \frac{1}{m}\int\sinh(mx)\,dx\right) \qquad \begin{bmatrix} U = x, & dV = \cosh(mx)\,dx, \\ dU = dx & V = \frac{1}{m}\sinh(mx)\end{bmatrix}$$
$$= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m^2}x\sinh(mx) + \frac{2}{m^3}\cosh(mx) + C$$

55. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1 \quad \Rightarrow$

$$\int \frac{dx}{x+4+4\sqrt{x+1}} = \int \frac{2u\,du}{u^2+3+4u} = \int \left[\frac{-1}{u+1} + \frac{3}{u+3}\right] du$$
$$= 3\ln|u+3| - \ln|u+1| + C = 3\ln\!\left(\sqrt{x+1}+3\right) - \ln\!\left(\sqrt{x+1}+1\right) + C$$

57. Let $u = \sqrt[3]{x+c}$. Then $x = u^3 - c \quad \Rightarrow$

$$\textstyle\int x\,\sqrt[3]{x+c}\,dx = \int \left(u^3 - c\right)u \cdot 3u^2\,du = 3\int \left(u^6 - cu^3\right)du = \frac{3}{7}u^7 - \frac{3}{4}cu^4 + C$$
$$= \tfrac{3}{7}(x+c)^{7/3} - \tfrac{3}{4}c(x+c)^{4/3} + C$$

59. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \quad \Rightarrow$

$$\int \frac{dx}{e^{3x}-e^x} = \int \frac{du/u}{u^3-u} = \int \frac{du}{(u-1)u^2(u+1)} = \int \left[\frac{1/2}{u-1} - \frac{1}{u^2} - \frac{1/2}{u+1}\right] du$$
$$= \frac{1}{u} + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = e^{-x} + \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C$$

61. Let $u = x^5$. Then $du = 5x^4\,dx \quad \Rightarrow$

$$\int \frac{x^4\,dx}{x^{10}+16} = \int \frac{\frac{1}{5}\,du}{u^2+16} = \tfrac{1}{5}\cdot\tfrac{1}{4}\tan^{-1}\!\left(\tfrac{1}{4}u\right) + C = \tfrac{1}{20}\tan^{-1}\!\left(\tfrac{1}{4}x^5\right) + C.$$

63. Let $y = \sqrt{x}$ so that $dy = \dfrac{1}{2\sqrt{x}}\,dx \quad \Rightarrow \quad dx = 2\sqrt{x}\,dy = 2y\,dy$. Then

$$\int \sqrt{x}\,e^{\sqrt{x}}\,dx = \int ye^y(2y\,dy) = \int 2y^2e^y\,dy \qquad \begin{bmatrix} u = 2y^2, & dv = e^y\,dy, \\ du = 4y\,dy & v = e^y \end{bmatrix}$$
$$= 2y^2e^y - \int 4ye^y\,dy \qquad \begin{bmatrix} U = 4y, & dV = e^y\,dy, \\ dU = 4\,dy & V = e^y \end{bmatrix}$$
$$= 2y^2e^y - \left(4ye^y - \textstyle\int 4e^y\,dy\right) = 2y^2e^y - 4ye^y + 4e^y + C$$
$$= 2(y^2 - 2y + 2)e^y + C = 2(x - 2\sqrt{x} + 2)\,e^{\sqrt{x}} + C$$

65. $$\int \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \int \left(\frac{1}{\sqrt{x+1}+\sqrt{x}} \cdot \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\right) dx = \int \left(\sqrt{x+1} - \sqrt{x}\right) dx$$
$$= \tfrac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C$$

67. Let $u = \sqrt{t}$. Then $du = dt/\left(2\sqrt{t}\,\right) \quad \Rightarrow$

$$\int_1^3 \frac{\arctan\sqrt{t}}{\sqrt{t}}\,dt = \int_1^{\sqrt{3}} \tan^{-1}u\,(2\,du) = 2\left[u\tan^{-1}u - \tfrac{1}{2}\ln(1+u^2)\right]_1^{\sqrt{3}} \qquad \text{[Example 5 in Section 7.1]}$$
$$= 2\left[\left(\sqrt{3}\tan^{-1}\sqrt{3} - \tfrac{1}{2}\ln 4\right) - \left(\tan^{-1}1 - \tfrac{1}{2}\ln 2\right)\right]$$
$$= 2\left[\left(\sqrt{3}\cdot\tfrac{\pi}{3} - \ln 2\right) - \left(\tfrac{\pi}{4} - \tfrac{1}{2}\ln 2\right)\right] = \tfrac{2}{3}\sqrt{3}\,\pi - \tfrac{1}{2}\pi - \ln 2$$

69. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \quad \Rightarrow$

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{u^2}{1+u}\,\frac{du}{u} = \int \frac{u}{1+u}\,du = \int \left(1 - \frac{1}{1+u}\right) du$$
$$= u - \ln|1+u| + C = e^x - \ln(1+e^x) + C$$

71. $\dfrac{x}{x^4+4x^2+3}=\dfrac{x}{(x^2+3)(x^2+1)}=\dfrac{Ax+B}{x^2+3}+\dfrac{Cx+D}{x^2+1} \quad\Rightarrow$

$$x=(Ax+B)\left(x^2+1\right)+(Cx+D)\left(x^2+3\right)=\left(Ax^3+Bx^2+Ax+B\right)+\left(Cx^3+Dx^2+3Cx+3D\right)$$
$$=(A+C)x^3+(B+D)x^2+(A+3C)x+(B+3D) \quad\Rightarrow$$

$A+C=0$, $B+D=0$, $A+3C=1$, $B+3D=0 \quad\Rightarrow\quad A=-\frac{1}{2}$, $C=\frac{1}{2}$, $B=0$, $D=0$. Thus,

$$\int\frac{x}{x^4+4x^2+3}\,dx=\int\left(\frac{-\frac{1}{2}x}{x^2+3}+\frac{\frac{1}{2}x}{x^2+1}\right)dx$$
$$=-\tfrac{1}{4}\ln\left(x^2+3\right)+\tfrac{1}{4}\ln\left(x^2+1\right)+C \quad\text{or}\quad \frac{1}{4}\ln\left(\frac{x^2+1}{x^2+3}\right)+C$$

73. $\dfrac{1}{(x-2)(x^2+4)}=\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+4} \quad\Rightarrow$

$1=A\left(x^2+4\right)+(Bx+C)(x-2)=(A+B)x^2+(C-2B)x+(4A-2C)$. So $0=A+B=C-2B$, $1=4A-2C$. Setting $x=2$ gives $A=\frac{1}{8} \quad\Rightarrow\quad B=-\frac{1}{8}$ and $C=-\frac{1}{4}$. So

$$\int\frac{1}{(x-2)(x^2+4)}\,dx=\int\left(\frac{\frac{1}{8}}{x-2}+\frac{-\frac{1}{8}x-\frac{1}{4}}{x^2+4}\right)dx=\frac{1}{8}\int\frac{dx}{x-2}-\frac{1}{16}\int\frac{2x\,dx}{x^2+4}-\frac{1}{4}\int\frac{dx}{x^2+4}$$
$$=\tfrac{1}{8}\ln|x-2|-\tfrac{1}{16}\ln\left(x^2+4\right)-\tfrac{1}{8}\tan^{-1}(x/2)+C$$

75. $\int\sin x\sin 2x\sin 3x\,dx=\int\sin x\cdot\frac{1}{2}[\cos(2x-3x)-\cos(2x+3x)]\,dx=\frac{1}{2}\int(\sin x\cos x-\sin x\cos 5x)\,dx$

$$=\tfrac{1}{4}\int\sin 2x\,dx-\tfrac{1}{2}\int\tfrac{1}{2}\,[\sin(x+5x)+\sin(x-5x)]\,dx$$
$$=-\tfrac{1}{8}\cos 2x-\tfrac{1}{4}\int(\sin 6x-\sin 4x)\,dx=-\tfrac{1}{8}\cos 2x+\tfrac{1}{24}\cos 6x-\tfrac{1}{16}\cos 4x+C$$

77. Let $u=x^{3/2}$ so that $u^2=x^3$ and $du=\frac{3}{2}x^{1/2}\,dx \quad\Rightarrow\quad \sqrt{x}\,dx=\frac{2}{3}\,du$. Then

$$\int\frac{\sqrt{x}}{1+x^3}\,dx=\int\frac{\frac{2}{3}}{1+u^2}du=\frac{2}{3}\tan^{-1}u+C=\frac{2}{3}\tan^{-1}(x^{3/2})+C.$$

79. Let $u=x$, $dv=\sin^2x\cos x\,dx \quad\Rightarrow\quad du=dx$, $v=\frac{1}{3}\sin^3x$. Then

$$\int x\sin^2x\,\cos x\,dx=\tfrac{1}{3}x\sin^3x-\int\tfrac{1}{3}\sin^3x\,dx=\tfrac{1}{3}x\sin^3x-\tfrac{1}{3}\int\left(1-\cos^2x\right)\sin x\,dx$$
$$=\frac{1}{3}x\sin^3x+\frac{1}{3}\int\left(1-y^2\right)dy \qquad \begin{bmatrix} y=\cos x, \\ dy=-\sin x\,dx\end{bmatrix}$$
$$=\tfrac{1}{3}x\sin^3x+\tfrac{1}{3}y-\tfrac{1}{9}y^3+C=\tfrac{1}{3}x\sin^3x+\tfrac{1}{3}\cos x-\tfrac{1}{9}\cos^3x+C$$

81. The function $y=2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\int(2x^2+1)e^{x^2}\,dx=\int 2x^2e^{x^2}dx+\int e^{x^2}dx=\int x\left(2xe^{x^2}\right)dx+\int e^{x^2}dx$$
$$=xe^{x^2}-\int e^{x^2}dx+\int e^{x^2}dx \quad \begin{bmatrix} u=x, & dv=2xe^{x^2}dx, \\ du=dx & v=e^{x^2}\end{bmatrix} \quad =xe^{x^2}+C$$